Harmonic manifolds with some specific volume densities

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Abstract

We show that noncompact simply connected harmonic manifolds with volume density $\Theta_p(r) = \sinh^{n-1} r$ is isometric to the real hyperbolic space and noncompact simply connected Kähler harmonic manifold with volume density $\Theta_p(r) = \sinh^{2n-1} r \cosh r$ is isometric to the complex hyperbolic space. A similar result is also proved for Quaternionic Kähler manifolds. Using our methods we get an alternative proof, without appealing to the powerful Cheeger-Gromoll splitting theorem, of the fact that every Ricci flat harmonic manifold is isometric to the euclidean space. Finally a rigidity result for real hyperbolic space is presented.

1 Introduction

Let (M, g) be a Riemannian manifold and let $p \in M$. Consider a normal coordinate neighbourhood U around p. Let $\omega_p = \sqrt{|\det(g_{ij})|}$ be the volume density function of M in U. We say that M is a harmonic manifold if ω_p is a function of the geodesic distance $r(p, .)$ alone. If (r, ϕ) is a polar coordinate system around p then the density becomes $\Theta_p = r^{n-1}\omega_p$. So M is harmonic if Θ_p is a function of r alone and hence can be written as $\Theta_p(r)$. Moreover $\Theta_p(r)$ is independent of the point p [1].

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Rank one symmetric spaces are harmonic as can be easily seen from their density function. Besides these there were no known examples of harmonic spaces. Moreover Lichnerowicz proved that upto dimension 4 harmonic spaces are in fact rank one symmetric. This led to the Lichnerowicz conjecture which asserts that *Every harmonic space is rank one symmetric*. It should be noted that even higher rank symmetric spaces are not harmonic.

Let (M, g) be a harmonic space. The well known Ledger's formula [1] (see pp. 161) gives

$$\omega_p''(r)|_{r=0} \;=\; -\frac{1}{3}\, Ricci_p$$

Hence for harmonic manifolds, since ω_p is a function of r alone, the ricci curvature is a constant, i.e harmonic spaces are Einstein. Let $Ricci(M) = k$. There arise three cases.

1. $k > 0$. In this case, by Myers-Bonnet theorem, M is compact with finite fundamental group and Szabo [7] proved that compact harmonic manifolds with finite fundamental group are rank one symmetric, thus settling the Lichnerowicz conjecture.

2. $k = 0$. Here one appeals to the powerful Cheeger-Gromoll splitting theorem to conclude that M is isometric to the euclidean space, i.e *Ricci flat harmonic mmanifolds are flat.*

3. $k < 0$. The Lichnerowicz conjecture is not true in this case. E. Damek and F. Ricci [3] constructed a family of nonsymmetric harmonic spaces. These spaces are called the NA spaces. In this family there are harmonic manifolds with same density function as that of the quaternionic hyperbolic space. All these spaces are homogeneous. Presently it is not known whether there are nonhomogeneous harmonic spaces. So it seems that the classification of harmonic spaces can be achieved only upto the determination of all density functions, i.e density classification of harmonic spaces. So Szabo [8] asked the following question.
   Question. Which harmonic spaces are determined by their density functions? i.e, which harmonic spaces are density equivalent?

We answer this question for three specific cases, namely that of the real, complex and

Quaternionic hyperbolic spaces.

Theorem 1 *Let (M,g) be a non-compact simply connected harmonic space with density function $\Theta_p(r) = \sinh^{n-1} r$, then (M,g) is isometric to the real hyperbolic space.*

Theorem 2 *Let (M,g) be a non-compact simply connected Kähler harmonic manifold with density funtion $\Theta_p(r) = \sinh^{2n-1} r \cosh r$, then M is isometric to the complex hyperbolic space.*

Theorem 3 *Let (M,g) be a noncompact simply connected Quaternionic Kähler harmonic manifold with volume density $\Theta_p(r) = \sinh^{4n-1} r \cosh^3 r$, then M is isometric to the quaternionic hyperbolic space.*

Theorems 1 and 2 explain the lack of examples of nonsymmetric (Kähler) harmonic spaces with same density function as that of the real (complex) hyperbolic space. Using the same methods we also give an alternative proof, without appealing to the powerful Cheeger-Gromoll splitting theorem, of the fact that every ricci flat harmonic manifold is isometric to the euclidean space,i.e,

Theorem 4 *Every simply connected ricci flat harmonic manifold is flat, i.e it is isometric to the euclidean space.*

In the next section we give the proofs of the above theorems. In the last section we give a rigidity result for the real hyperbolic space. The authors would like to thank their colleague G. Santhanam for discussions on the subject.

2 Density equivalent spaces

Proof of Theorem 1

Let $p \in M$ and $S_{p,R}$ be the distance sphere around p of radius R. From the well known Ledgers formula [1] (see pp. 161) we have

$$\nabla_m \nabla_m \omega_p = -2/3 Ricci(m,m) \;\; for \;\; p \in M \;\; and \;\; m \in T_p(M).$$

hence we get $Ricci(g) = -(n-1)$. Again $\Theta_p(r) = \sinh^{n-1} r$ gives that the mean curvature $\sigma_p(R)$ of $S_{p,R}$ is,

$$\sigma_{p,R} = \Theta_p'/\Theta p = (n-1)\coth r.$$

Let L be the second fundamental form of $S_{p,R}$, then $TraceL = (n-1)\coth r$. Using the Riccati equation $L' + L^2 + R\,(\gamma'\,.\,)\,\gamma' = 0$ we get

$$TrL' + TrL^2 + Ricci = 0 \;\; i.e$$

$$TrL^2 = (n-1)\coth^2 r = 1/(n-1)(TrL)^2.$$

For any linear map L we know that $TrL^2 \geq 1/(n-1)(TrL)^2$ and equality holds iff L is a scalar operator. So L is a scalar operator, i.e $L = \coth r Id$ which shows that M is of constant sectional curvature -1, hence M is isometric to the real hyperbolic space. ■

Proof of Theorem 2

Ledgers formula gives $Ricci(g) = -(2n+2)$. Let $\gamma(t)$ be any geodesic. Let J be the complex structure on M and I be the index form of M. Let $T > 0$ be a real number. Let $E_2, E_3, ..., E_{2n}$ be unit orthogonal parallel fields along γ, normal to $\gamma'(t)$ with $E_2(t) = J\,\gamma'(t)$. Let $J_i\,(t)$ be jacobi fields along $\gamma(t)$ such that

$$J_i(0) = 0, \; and \; J_i(T) = E_i(T)$$

Let

$$X_2(t) = \frac{\sinh 2t}{\sinh 2T}\,E_2(t)$$

and

$$X_i(t) = \frac{\sinh t}{\sinh T} E_i(t), \; i = 3, \cdots, 2n.$$

be vector fields along $\gamma(t)$. Note that $J_i(0) = X_i(0)$ and $J_i(T) = X_i(T)$. Since there are no conjugate points along γ, we get $I(J_i, J_i) \leq I(X_i, X_i)$. Summing we get

$$\sum_{i=2}^{2n} I(J_i, J_i) \leq \sum_{i=2}^{2n} I(X_i, X_i)$$

But $\sum_{i=2}^{2n} I(J_i, J_i) = \frac{\Theta_p'(T)}{\Theta_p(T)}$. A simple calculation gives

$$I(X_2, X_2) = \frac{1}{2\sinh^2 2T}(\sinh 4T + 4T) - \int_0^T \frac{\sinh^2 2t}{\sinh^2 2T} H(\gamma'(t))\,dt$$

$$I(X_i, X_i) \;=\; \frac{2T + \sinh 2T}{4 \sinh^2 T} \;-\; \int_0^T \frac{\sinh^2\; t}{\sinh^2\; T}\; K(\gamma'(t), E_i(t))\, dt\;,\; i \;=\; 3, \cdots, 2n$$

Here $K(x, y)$ is the sectional curvature of the plane spanned by x, y, and $H(\gamma'(t)) = K(\gamma'(t), J\gamma'(t))$ is the holomorphic sectional curvature. Hence after simplifying and using Ricci = -2(n+1)

$$\sum_{i=2}^{2n} I(X_i, X_i) \;=\; A(T) \;+\; \int_0^T \; B(t)\; H(\gamma'(t))\; dt$$

where

$$A(T) \;=\; \frac{\sinh 4T \;+\; 4T}{2\; \sinh^2\; 2T} \;+\; \frac{1}{4\; \sinh^2\; T}\,(4n\; \sinh 2T \;-\; 8T)$$

and

$$B(t) \;=\; \frac{\sinh^2\; t}{\sinh^2\; T} \;-\; \frac{\sinh^2\; 2t}{\sinh^2\; 2T}$$

Hence one gets

$$\frac{\Theta_p'(T)}{\Theta_p(T)} \;\leq\; (A(T) \;-\; 4\, C(T)) \;+\; \int_0^T \; B(t)\, (H(\gamma'(t)) \;+\; 4)\, dt$$

where $C(T) \;=\; \int_0^T \; B(t)\; dt$.
Note that $B(t) \;\geq\; 0, t \in [0, T]$ and $A(T) \;-\; 4\, C(T) \;=\; \frac{\Theta_p'(T)}{\Theta_p(T)}$. Hence it follows that

$$\int_0^T \; B(t)\, (H(\gamma'(t)) \;+\; 4)\, dt \;\geq\; 0$$

which in turn gives $H(v) \;\geq\; -4$ for all unit vectors v. An algebraic calculation [5] yields

$$\int_{U_pM} H(v)\, dv \;=\; \frac{Vol(U_pM)}{n(n+1)}\, Scal_pM$$

where Scal is the scalar curvature of M. In our case it is $-4n(n+1)$. So we get

$$\int_{U_pM} H(v)\, dv \;=\; -4\, Vol(U_pM)$$

Combined with the conclusion $H(v) \geq -4$ we get $H \;\equiv\; -4$, and M is isometric to the complex hyperbolic space. ∎

Proof of Theorem 3

Consider a chart $(U, p), p \in M$ with two almost complex J_1, J_2 such that the Levi-Civita derivatives of J_1, J_2 are linear combinations of J_1, J_2 and $J_3 = J_1 J_2$. Let $\gamma(t)$ be a geodesic starting at p. Choose $T > 0$ such that $\gamma[0, T] \subset U$. Let I be the index form on γ. Let $E_2(t), \cdots, E_{4n}(t)$ be unit orthogonal parallel fields along γ such that $E_2(t), E_3(t), E_4(t)$ belong to the three dimensional subbundle spanned by $J_1\gamma'(t), J_2\gamma'(t)$ and $J_3\gamma'(t)$.

Now $Ricci(M) = -(4n+8)$ as can be seen from the Ledgers formula. The choice of J_1, J_2, J_3 shows that the four dimensional space spanned by $\{\gamma'(t), J_1\gamma'(t), J_2\gamma'(t), J_3\gamma'(t)$ is a quaternionic line parallel along γ which has an $Sp(1)$ action. Therefore we get a family of almost complex structures $J_1(t), J_2(t), J_3(t)$ along γ such that

$$E_i(t) \;=\; J_i(t)\,,\; J_i(0) \;=\; J_i, \text{ for } i \;=\; 1,\, 2,\, 3$$

The rest of the proof is similar to that of theorem 2, so we shall be brief. Take

$$X_i(t) \;=\; \frac{\sinh 2t}{\sinh 2T}\, E_i(t)\,,\; i \;=\; 2,\, 3,\, 4$$

and

$$X_i(t) \;=\; \frac{\sinh t}{\sinh T}\, E_i(t)\,,\; i \;=\; 5,\, \cdots,\, 4n$$

Let $J_i(t)$ be jacobi fields along $\gamma(t)$ such that $J_i(0) = X_i(0)$ and $J_i(T) = X_i(T)$. Since there are no conjugate points along $\gamma(t)$ the following inequality holds

$$\sum_{i=2}^{4n} I(J_i, J_i) \;\le\; \sum_{i=2}^{4n} \; I(X_i, X_i)$$

and equality holds iff $X_i(t) = J_i(t)\ \forall\, i$. Now

$$\sum_{2}^{4n} I(X_i, X_i) \;=\; (A(T) \;-\; 12\, C(T)) \;+\; \int_0^T \, B(t) \left(\sum_{2}^{4} K(\gamma'(t), E_i(t)) \;+\; 12 \right) dt$$

where

$$A(T) = \frac{3\,(\sinh 4T \;+\; 4T)}{2\, \sinh^2 2T} \;+\; \frac{((8n \;+\; 4)\, \sinh 2T \;-\; 24T)}{4\, \sinh^2 T}$$

$$B(t) \;=\; \frac{\sinh^2\, t}{\sinh^2\, T} \;-\; \frac{\sinh^2\, 2t}{\sinh^2\, 2T} \;>\; 0\;,\; t \;\in\; [0, T]$$

and

$$C(T) \;=\; \int_0^T \; B(t)\; dt$$

and $K(x,y)$ stands for the sectional curvature of the plane spanned by vectors x and y. Using $\sum_{i=2}^{2n} I(J_i, J_i) \;=\; \frac{\Theta_p'(T)}{\Theta_p(T)}$ and $E_i(t) = J_i(t)$ we get

$$\frac{\Theta_p'(T)}{\Theta_p(T)} \leq (A(T) \;-\; 12\, C(T)) \;+\; \int_0^T \; B(t) \left(\sum_2^4 \, K(\gamma'(t), E_i(t)) \;+\; 12 \right) \, dt$$

Now we use the following relation between the components of the curvature tensor [2],

$$\sum_1^3 K\,(X\,,\; J_i(t)X) \;=\; \frac{3}{n+2}\, Ricci(M) \;=\; -12$$

to finally get

$$\frac{\Theta_p'(T)}{\Theta_p(T)} \leq \; A(T) \;-\; 12\, C(T)$$

and equality holds iff X_i is a jacobi field for all i. A simple computation verifies that in fact equality holds. Thus $X_i(t) = J_i(t)$ for all i. Since the point p is arbitrary and γ is any geodesic starting at p, M is isometric to the quaternionic hyperbolic space. ■

Proof of Theorem 4

Let γ be a geodesic ray in M. Let F_γ be the Busemann function relative to the geodesic ray γ. Take $X_i(t) \;=\; \frac{t}{T}\, E_i(t)$ in the proof of Theorem 2 to get

$$\frac{\Theta_p'(T)}{\Theta_p(T)} \leq \frac{(n-1)}{T}$$

But convexity of balls gives $\frac{\Theta_p'(T)}{\Theta_p(T)} \geq 0$. Therefore

$$\frac{\Theta_p'(T)}{\Theta_p(T)} \;\to 0 \; as \; T \;\to\; \infty$$

i.e, $Tr L \;=\; 0$ where L is the *second fundamental form* of the horospheres determined by F_γ. Since Ricci $= 0$, Riccati equation gives $Tr\, L' \,+\, Tr\,(L^2) \;=\; 0$, but $Tr\, L' \;=\; 0$, hence $Tr\,(L^2) \;=\; 0$. Symmetry of L now gives that $L \;=\; 0$. This shows that the sectional curvature $K(\gamma'(t), .) = 0$. But γ is a arbitrary geodesic, hence $K(M) \equiv 0$ and the proof is complete. ■

3 Rigidity of H^n

In this section we prove that the real hyperbolic space is rigid among all harmonic spaces. More precisely we show that, if a harmonic space is asymptotically density equivalent to the real hyperbolic space then they are actually isometric.

Let (M,g) be a non-compact harmonic manifold. Normalising the metric on (M,g), let us assume that $Ricci(M,g) = -(n-1)$. The Bishop-Gromov volume comparison theorem [4] (pp. 144-147) or [6] (pp. 140) gives the density function to be

$$\Theta_p(r) = \alpha(r)\sinh^{n-1} r,$$

where $\alpha(r)$ satisfies

$$0 \le \alpha(r) \le 1\,;\, \alpha(0) = 1\,,\, \alpha'(0) = 0 \,\text{and}\, \alpha'(r) \le 0.$$

Hence $\alpha(r)$ is a decreasing function. Two cases arise:
1. $\lim_{r\to\infty} \alpha(r) = 0$ and
2. $\lim_{r\to\infty} \alpha(r) = c$ for some constant $c > 0$.
In the second case the density of M is asymptotically same as that of the real hyperbolic case. In this case we show that $c = 1$ and M is in fact isometric to the real hyperbolic space.

Theorem 5 *Let (M,g) be a non-compact harmonic manifold with $\lim_{r\to\infty}\alpha(r) = c > 0$. Then $c = 1$ and M is isometric to the real hyperbolic space of constant sectional curvature -1.*

Proof. Let γ be a geodesic ray in M. Let F_γ be the Busemann function of γ. Now

$$\lim_{r\to\infty} \alpha(r) = c > 0$$

implies that the density of the horospheres determined by F_γ is

$$\Theta\,(r) = c\,\sinh^{n-1} r$$

Hence the mean curvature of these horospheres is

$$\sigma_p\,(r) = (n-1)\coth r = Tr\,L$$

where L is the *second fundamental form* of the horospheres. Now the Riccati equation combined with $Ricci = -(n-1)$ gives

$$Tr\,(L^2) = (n-1)\coth^2 r$$

but

$$Tr\,L = (n-1)\coth r$$

Hence by Cauchy-Schwartz inequality one gets that $L = (\coth r)\,Id$. Thus M is isometric to the real hyperbolic space of constant curvature -1. ■

4 Remarks

For the complex hyperbolic space an easy calculation shows that $\alpha(r) \to 0$ as $r \to \infty$. Now assume that M is Kähler harmonic space. Normalize the metric so that $Ricci(M) = -(2n+2)$. Again applying the Bishop-Gromov comparison theorem one sees that the volume density of M is

$$\Theta(r) \;=\; \beta(r)\, \sinh^{2n-1} r\, \cosh r$$

where $\beta(r)$ satisfies the same properties as that of $\alpha(r)$. The following question is natural.

Question. If $\beta(r) \to b(>0)$ as $r \to \infty$, is M isometric to the complex hyperbolic space.

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